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# On 1-qubit channels

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**Abstract**

The entropy $H_T(\rho)$ of a state with respect to a channel $T$ and the Holevo capacity of the channel require the solution of difficult variational problems. For a class of 1-qubit channels, which contains all the extremal ones, the problem can be significantly simplified by attaching a unique Hermitian antilinear operator $\vartheta$ to every channel of the considered class. The channel's concurrence $C_T$ can be expressed by $\vartheta$ and turns out to be a flat roof. This allows to write down an explicit expression for $H_T$. Its maximum would give the Holevo (one-shot) capacity.

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## 1. Introduction

Given two Hilbert spaces, $\mathcal{H}^{\mathrm{in}}$ and $\mathcal{H}^{\mathrm{out}}$, of finite dimension, a quantum channel is a completely positive and trace preserving linear map,

$$T : \rho^{\mathrm{in}} \longrightarrow \rho^{\mathrm{out}} := T(\rho^{\mathrm{in}}) \tag{1}$$

from the operators of $\mathcal{H}^{\mathrm{in}}$ into those of $\mathcal{H}^{\mathrm{out}}$. For shorter notation, we skip the in and out superscripts mostly. The *rank* of $T$ is the maximal rank within all output density operators $T(\rho)$.

An important device to estimate how effectively a channel works is the quantity

$$H_T(\rho) = \max \sum p_j S[T(\rho) \parallel T(\rho_j)]. \tag{2}$$

In this variational problem one has to compare all convex decompositions

$$\rho = \sum p_j \rho_j \qquad p_j \geqslant 0 \tag{3}$$

of the input density operator $\rho$ into other input density operators $\rho_j$, and $S(\cdot \parallel \cdot)$ abbreviates the relative entropy. According to Holevo [1], the quantity (2) can be interpreted as the maximum of mutual information between the input and the output of $T$ for a given ensemble average $\rho$. In Benatti [2] it is identified with the maximal accessible information for all quantum sources with ensemble average $\rho$. In Schumacher and Westmoreland [3], where the problem is considered in an even more general context, this maximum is denoted by $\chi^*$, and an ensemble saturating (2) is called an *optimal signal ensemble*. One gets the *Holevo* or *one-shot capacity* by

$$\mathbf{C}(T) = \max_\rho H_T(\rho).$$

On the other hand, (2) is a decisive tool for the construction of the CNT-entropy of Connes, Narnhofer and Thirring [4]. In this context it is called *entropy of* $\rho^{\text{in}}$ *with respect to the channel T*, an appropriate generalization of the *entropy of* $\rho$ *with respect to subalgebra* [5]. (I apologise for the change in notation: in [4] and [6] the quantity $H_T(\rho)$ has been called $H_\rho(T)$.) By the von Neumann entropy

$$S_T(\rho) = S[T(\rho)]$$

of $T(\rho)$, (2) can be rewritten as

$$H_T(\rho) = S_T(\rho) - \min \sum p_j S_T(\rho_j).$$

The concavity of $S$ allows to restrict the convex decompositions (3) to the extremal ones, i.e. to those consisting of pure input states only. This simple observation implies that we may write

$$H_T(\rho) = S_T(\rho) - E_T(\rho) \tag{4}$$

where

$$E_T(\rho) = \min p_j S_T(\pi_j) \qquad \pi_j \text{ pure} \tag{5}$$

under the condition $\rho = \sum p_j \pi_j$. The simplification is not only by the restriction of the original variation to decompositions with pure states. $E_T$ enjoys in addition the *roof property*: if one can find an optimal decomposition of $\rho$ into pure states saturating (5), then $E_T$ is convexly linear on the convex subset generated by the pure input states $\pi_j$ (provided $p_j > 0$). Moreover, $E_T$ is convexly linear on the convex set generated by the union of *all* those pure input states $\pi$ which can occur in any optimal decomposition of a given input density operator $\rho$; see [6] or [8] for more details.

For the 1-qubit channel, which substitutes the off-diagonal elements of $\rho$ by zeros, the first published computation of $E_T$ and $H_T$ that I know of is by Levitin [7]. He explicitly points to the constancy of $E_T$ along the straight lines of density operators parallel to the 3-axis with fixed off-diagonal elements. But before knowing of Levitin's work I had seen this surprising feature by a computer program of R F Werner (Vienna 1994). An instructive example, how this observation can be used to compute parts of $E_T$ of the same problem but for rank three, is in Benatti *et al* [9, 10].

On the other hand, if $T$ is the relative trace to the states of one part in a bipartite system, $E_T$ is the *entanglement of formation* of Bennett *et al* [13]. These authors compute $E_T$ for Werner states [12] in $2 \times 2$ dimensions. Higher-dimensional Werner states are treated in Terhal and Vollbrecht [14] and Vollbrecht and Werner [15]. As a heuristic guide, one may think of $E_T(\rho)$ as a measure of entanglement of $\rho$ with respect to an arbitrary channel $T$.

The convex roof extension, $E_T$, is the largest convex function on the input density operators coinciding with $S_T$ for pure states $\pi = \pi^{\text{in}}$. As $-E_T$ is concave, the computation of $H_T$ can be paraphrased as follows:

*Add to $S_T$ the smallest concave function of the input states such that the sum vanishes at pure input states. One obtains $H_T$.*

Now I pass to a quite different topic. In certain cases one can effectively compute channel characteristics by antilinear operators. An antilinear operator $\vartheta$ acts on kets according to

$$\vartheta \sum a_j |j\rangle = \sum a_j^* \vartheta |j\rangle.$$

With respect to a basis, $\vartheta$ is completely described by its matrix elements $\langle j|\vartheta|k\rangle$ and, to distinguish its matrix representation from the linear situation, I add an index *anti* to it. Hence, in two dimensions, let us write

$$\vartheta = \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix}_{\text{anti}}$$

for the matrix representation based on $|0\rangle$ and $|1\rangle$. As a merit we can easily compute matrix products. For instance,

$$\{\alpha_{jk}\}_{\text{anti}} \cdot \{a_{li}\} = \left\{ \sum_k \alpha_{jk} a_{ki}^* \right\}_{\text{anti}}$$

represents the product of an antilinear and a linear operator, which is again antilinear. The Hermitian adjoint, $\vartheta^\dagger$, of an antilinear $\vartheta$ is again antilinear and defined by the rule

$$\langle j|\vartheta^\dagger|k\rangle = \langle k|\vartheta|j\rangle.$$

The Hermitian conjugate thus changed the matrix elements of an antilinear operator from $\alpha_{jk}$ to $\alpha_{kj}$. It follows that $\vartheta \to \vartheta^\dagger$ is a *linear* operation for antilinear operators, quite in contrast to the linear case. We shall mostly need Hermitian (i.e. selfadjoint) antilinear operators. The matrix entries for these operators are characterized by the symmetry condition $\alpha_{jk} = \alpha_{kj}$, and by nothing else.

There are some warnings concerning the use of antilinear operators and maps. Two of them are: (1) Do not apply them to bras, i.e. from right to left: an expression like $\langle y| = \langle x|\vartheta$ is ill-defined. (2) One cannot tensor an antilinear operator with a linear one.

Next, $\vartheta\vartheta^\dagger$ and $\vartheta^\dagger\vartheta$ are positive linear operators with equal eigenvalues. The eigenvalues of an antilinear operator itself, however, fill some circles; see Wigner [11] for more details. Therefore, the determinant is defined only up to a phase factor. The trace is undefined for antilinear operators.

## 2. Rank-two channels

Let us now assume $\mathcal{H}^{\text{out}} = 2$. Then there is only one free variable on which $T(\rho)$ depends. This fact has been used already in [13] and, with a very remarkable result, in Hill and Wootters [16] and Wootters [17] to compute the entanglement of formation in the two-quibit case.

Following [13], the first issue is to start with a suitable expression for $S_T$. With

$$h(x) = -x \ln_2 x - (1-x) \ln_2(1-x)$$

one defines

$$f(y) := h\left(\frac{1+\sqrt{1-y^2}}{2}\right) \tag{6}$$

which is increasingly monotone in $0 \leqslant y \leqslant 1$ from 0 to 1, and convex in $-1 \leqslant y \leqslant 1$. (One checks that the first derivative, $f'$, is increasing.) For a $2 \times 2$ density operator $\omega$ with eigenvalues $\mu_1 \geqslant \mu_2$ one gets

$$S(\omega) = h(\mu_1) = f(y) \qquad y = 2\sqrt{\det \omega} \tag{7}$$

which can be seen from

$$1 - y^2 = (\mu_1 + \mu_2)^2 - 4\mu_1\mu_2 = (\mu_1 - \mu_2)^2.$$

It follows that

$$S_T(\rho) = f[2\sqrt{\det T(\rho)}]. \tag{8}$$

Our next task is to define the convex roof $C_T$ to be the largest convex function on $\mathcal{H}^{\text{in}}$ which coincides for pure input states $\pi$ with $\sqrt{\det T(\pi)}$. The letter $C$ and the name *concurrence of* $T$ for $C_T$ I borrowed from [13] and from [17]. To give an equation,

$$C_T(\rho) = \min \sum p_j \sqrt{\det T(\pi_j)} \tag{9}$$

the minimum is running through all convexly linear decompositions $\sum p_j \pi_j$ of $\rho$ with pure input states. As a matter of fact, one cannot beat this minimum in allowing the $\pi_j$ to become mixed. This is due to the concavity of $\sqrt{\det \omega}$ in two dimensions. (In the language of convex analysis: the convex hull of a concave function is a roof, see the appendix of [6].) As a by-product

$$C_T(\rho) \leqslant \sqrt{\det T(\rho)}.$$

The range of $C_T$ is from 0 to 1, and it is convex by definition. Because $f$ in (6) is convex and increasing, the function

$$\rho \longrightarrow f[2C_T(\rho)] \tag{10}$$

is a convex function which equals $S_T$ for pure states. Though $C_T$ is a roof, this is not sufficient for proving the equality of (10) with $E_T$. Why should a function of a roof remain a roof? There is no general reason for that. There exists, however, one special case not burdened with the mentioned difficulty: let us call $C_T$ *flat* if there is, for every $\rho$, an optimal pure state decomposition

$$\rho = \sum p_j \pi_j, \qquad C_T(\rho) = \sum p_j C_T(\pi_j)$$

such that

$$C_T(\pi_1) = C_T(\pi_2) = \cdots = C_T(\pi_j) = \cdots. \tag{11}$$

If this takes place, every $\rho$ is contained in a convex subset which is generated by pure input states, and on which the roof is not only linear but even constant.

Thus, if we would know the flatness of $C_T$, every function of it must be a roof, though not necessarily a convex one. But the convexity of (10) has been stated already. Altogether one arrives at:

**Lemma 1.** *If the roof $C_T$ is flat then*

$$E_T(\rho) = f[2C_T(\rho)]. \tag{12}$$

We are faced with two problems: how to compute $C_T$, and how to check whether it is a flat roof. The next aim is to give a large class of rank-two channels fulfilling the desired flatness condition.

## 3. 1-qubit channels of length two

Let $\mathcal{H}$ be of dimension two, and $T$ a quantum channel of the form

$$T(\rho) = A_\rho A^\dagger + B_\rho B^\dagger. \tag{13}$$

. The set of channels mapping the 1-qubit density operators into themselves is convex. Its structure is well described in King and Ruskai [18] and in Ruskai *et al* [19], where a complete list of all its extremal maps has been given. As shown in [19], every extremal 1-qubit channel has a representation (13). We may, for example, choose

$$A = \begin{pmatrix} a_{00} & 0 \\ 0 & a_{11} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & b_{01} \\ b_{10} & 0 \end{pmatrix} \tag{14}$$

To be trace preserving one has to have

$$|a_{00}|^2 + |b_{10}|^2 = |a_{11}|^2 + |b_{01}|^2 = 1.$$

According to [19], one can choose $A$ and $B$ in (14) with real entries to get all the extremal maps up to unitary equivalence. We are going to prove:

*For all quantum channels of the form (13) $C_T$ is flat, and there exist explicit expressions for $C_T$, $E_T$, and $H_T$.* One of the two key observations is:

**Theorem 2.** *Given a superoperator T as in (13), there is exactly one Hermitian antilinear operator, $\vartheta$, such that*

$$\det T(\pi) = \operatorname{tr} \pi(\vartheta \pi \vartheta) \tag{15}$$

*is true for all pure density operators $\pi$.*

**Proof.** The proof of the theorem goes in three steps. In the first two, both sides of (15) are computed. The last one is a comparison of the results.

Let $a_{jk}$, $j, k = 0, 1$, be the matrix elements of $A$ with respect to a reference basis. Accordingly let us write $B = \{b_{jk}\}$. The application of $A$ and $B$ to a vector $\{x_0, x_1\}$ is called $\{z_0, z_1\}$ and $\{w_0, w_1\}$ respectively. Hence

$$T\left(\begin{pmatrix} x_0x_0^* & x_0x_1^* \\ x_1x_0^* & x_1x_1^* \end{pmatrix}\right) = \begin{pmatrix} z_0z_0^* + w_0w_0^* & z_0z_1^* + w_0w_1^* \\ z_1z_0^* + w_1w_0^* & z_1z_1^* + w_1w_1^* \end{pmatrix}.$$

The determinant is given by

$$\det T\left(\begin{pmatrix} x_0x_0^* & x_0x_1^* \\ x_1x_0^* & x_1x_1^* \end{pmatrix}\right) = (z_0w_1 - z_1w_0)(z_0w_1 - z_1w_0)^*. \tag{16}$$

From $z_0 = a_{00}x_0 + a_{01}x_1$, $w_1 = b_{10}x_0 + b_{11}x_1$, and so on, we get the $w_j$ by using the coefficients $b_{jk}$. Hence

$$z_0w_1 - z_1w_0 = c_{00}x_0^2 + c_{11}x_1^2 + (c_{01} + c_{10})x_0x_1 \tag{17}$$

where

$$\begin{aligned} &c_{00} = a_{00}b_{10} - a_{10}b_{00} \qquad c_{11} = a_{01}b_{11} - a_{11}b_{01} \\ &c_{01} + c_{10} = a_{00}b_{11} + a_{01}b_{10} - a_{10}b_{01} - a_{11}b_{00}. \end{aligned} \tag{18}$$

Let us now consider step two of the proof. An antilinear operator, $\vartheta$, can be characterized by the entries of its matrix representation in a given reference basis. Let us denote $\vartheta$ and its Hermitian adjoint by

$$\vartheta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_{\text{anti}} \qquad \vartheta^\dagger = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}_{\text{anti}}. \tag{19}$$

For a general density operator, $\rho$, with entries $\rho_{jk}$ in the reference basis, one obtains

$$\vartheta^\dagger \rho \vartheta = \begin{pmatrix} \rho_{00}\alpha\alpha^* + \rho_{10}\alpha\gamma^* + \rho_{01}\gamma\alpha^* + \rho_{11}\gamma\gamma^* & \rho_{00}\alpha\beta^* + \rho_{10}\alpha\delta^* + \rho_{01}\gamma\beta^* + \rho_{11}\gamma\delta^* \\ \rho_{00}\beta\alpha^* + \rho_{10}\beta\gamma^* + \rho_{01}\delta\alpha^* + \rho_{11}\delta\gamma^* & \rho_{00}\beta\beta^* + \rho_{10}\beta\delta^* + \rho_{01}\delta\beta^* + \rho_{11}\delta\delta^* \end{pmatrix}.$$

It follows that

$$\vartheta^\dagger \begin{pmatrix} x_0x_0^* & x_0x_1^* \\ x_1x_0^* & x_1x_1^* \end{pmatrix} \vartheta = \begin{pmatrix} (\alpha x_0^* + \gamma x_1^*)(\alpha^* x_0 + \gamma^* x_1) & (\alpha x_0^* + \gamma x_1^*)(\beta^* x_0 + \delta^* x_1) \\ (\beta x_0^* + \delta x_1^*)(\alpha^* x_0 + \gamma^* x_1) & (\beta x_0^* + \delta x_1^*)(\beta^* x_0 + \delta^* x_1) \end{pmatrix}$$

and, finally,

$$\operatorname{tr} \begin{pmatrix} x_0x_0^* & x_0x_1^* \\ x_1x_0^* & x_1x_1^* \end{pmatrix} \vartheta^\dagger \begin{pmatrix} x_0x_0^* & x_0x_1^* \\ x_1x_0^* & x_1x_1^* \end{pmatrix} \vartheta = |x_0(\alpha^* x_0 + \gamma^* x_1) + x_1(\beta^* x_0 + \delta^* x_1)|^2. \tag{20}$$

Comparing with (18), the determinant of $T(\pi)$ is equal to the trace (20) if

$$\alpha^* = c_{00} \qquad \beta^* + \gamma^* = c_{01} + c_{10}, \qquad \delta^* = c_{11}. \tag{21}$$

With this choice we have

$$z_0 w_1 - w_1 z_0 = \langle\phi|\vartheta|\phi\rangle^*. \tag{22}$$

Now we impose Hermiticity. $\vartheta$ is Hermitian if and only if $\beta = \gamma$. We see from (21) that there is exactly one Hermitian antilinear $\vartheta$ with which (15) is satisfied. This proves the theorem. □

Before going ahead, let us write down $\vartheta$ for the subset of channels with Kraus operators (14).

Denoting the matrix entries as in (19), we get $\beta = \gamma = 0$ and

$$\alpha = a_{00}^* b_{10}^*, \quad \delta = -a_{11}^* b_{01}^* \tag{23}$$

To get the last piece of the puzzle I recall, as an adoption of [17], a definition of [8]. Define, for two general density operators $\omega_1$ and $\omega_2$,

$$C(\omega_1, \omega_2) := \max\left\{0 \quad \lambda_1 - \sum_{j>1} \lambda_j\right\}. \tag{24}$$

where the λs are the decreasingly ordered eigenvalues of

$$\left(\sqrt{\omega_1}\omega_2\sqrt{\omega_1}\right)^{1/2}.$$

If $\omega_1$ and $\omega_2$ are both of rank two, there are not more than two non-zero eigenvalues. This reduces (24) to $|\lambda_1 - \lambda_2|$, and one obtains the expression [8]

$$C(\omega_1, \omega_2)^2 = \operatorname{tr}\omega_1\omega_2 - 2\sqrt{\det\omega_1 \det\omega_2}. \tag{25}$$

There is a general feature of (24) which is proved in [8]:

**Theorem 3.** *Let $\vartheta$ be an antilinear Hermitian operator in an Hilbert space. The function*

$$\omega \longrightarrow C(\omega, \vartheta\omega\vartheta) \tag{26}$$

*is a flat convex roof on the set of density operators.*

Now, returning to our 1-qubit channels, let us look for the values of (26) for a pure state $\pi = |\phi\rangle\langle\phi|$. By (25) it is really easy to see that

$$C(\pi, \vartheta\pi\vartheta)^2 = \operatorname{tr}\pi\vartheta\pi\vartheta = |\langle\phi|\vartheta|\phi\rangle|^2. \tag{27}$$

By combining theorems 2 and 3, the structure of $E_T$ for the channels (13) becomes evident. By theorem 2 we find

$$\det T(\pi) = C(\pi, \vartheta\pi\vartheta)^2$$

and, finally,

$$C_T(\rho)^2 = C(\rho, \vartheta\rho\vartheta)^2 = \operatorname{tr}(\rho\vartheta\rho\vartheta) - 2\det\rho\sqrt{\det(\vartheta^2)} \tag{28}$$

$$E_T(\rho) = f[2C(\rho, \vartheta\rho\vartheta)] \tag{29}$$

and this is the solution of the variational problem we looked for.

*Examples.* For the channels with Kraus operators (13) the expression (28) can be made more explicit. In this case the matrix representation of $\vartheta$ is diagonal with entries (23). Hence

$$\vartheta\rho\vartheta = \begin{pmatrix} \rho_{00}\alpha\alpha^* & \rho_{10}\alpha\delta^* \\ \rho_{01}\delta\alpha^* & \rho_{11}\delta\delta^* \end{pmatrix}$$

$$\operatorname{tr}(\rho\vartheta\rho\vartheta) = \rho_{00}^2\alpha\alpha^* + \rho_{10}^2\alpha\delta^* + \rho_{01}^2\delta\alpha^* + \rho_{11}^2\delta\delta^*.$$

This we have to insert in (28), recalling that we have to subtract $\det \rho$ multiplied with twice the absolute value $|\alpha\delta|$ of $\alpha\delta$. We take a root of $\alpha\delta^*$ and choose its complex conjugate as the root of $\alpha^*\delta$. With this convention the following is unambiguous:

$$C_T^2 = (|\alpha|\rho_{00} - |\delta|\rho_{11})^2 + |\sqrt{\alpha\delta^*\rho_{01}} + \sqrt{\alpha^*\delta\rho_{10}}|^2. \tag{30}$$

First let us treat the degenerate case with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{t} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & \sqrt{1-t} \\ 0 & 0 \end{pmatrix} \tag{31}$$

and $1 \geqslant t > 0$. Then (30) reduces to

$$C_T(\rho) = \sqrt{t(1-t)}. \tag{32}$$

The foliation of a set of density operators induced by $C_T$ and $E_T$ is given by the intersections of the Bloch ball with the planes perpenticular to the 3-axis. $S_T$ is the von Neumann entropy of

$$T(\rho) = \begin{pmatrix} 1 - t\rho_{00} & \sqrt{t}\rho_{01} \\ \sqrt{t}\rho_{10} & t\rho_{11} \end{pmatrix}. \tag{33}$$

The determinant of $T(\rho)$, given $\rho_{11}$, is maximal for $\rho_{01} = 0$, and the same with $S_T$. Therefore, on a given leaf with constant $C_T$, the maximum of $S_T$ is $h(t\rho_{11})$. It follows that

$$H_T(\rho) \leqslant h(t\rho_{11}) - \tfrac{1}{2}h\Big(1 + \sqrt{1 - 4t(1-t)\rho_{11}^2}\Big)$$

on the plane containing the density operators with given $\rho_{11}$. Hence

$$\mathbf{C}(T) = \max_{0\leqslant r\leqslant 1}\big[h(rt) - \tfrac{1}{2}h\big(1 - \sqrt{1 - 4t(1-t)r^2}\big)\big]. \tag{34}$$

Smolin [20] has shown that the maximum is not achieved for orthogonal input states. (The first but more complicated example is by Fuchs [21].) Indeed, as long as $\rho_{11} \neq 1/2$, there are no pairs of orthogonal states in the leaves dictated by $C_T$.

Switching to the non-degenerate case, the leaves of constant concurrence $C_T$ are the intersection of straight lines with the Bloch ball. We get such a line by first fixing a plane of operators with constant diagonal entries. A second plane is obtained by constraining the off-diagonal entries to

$$\sqrt{\alpha\delta^*}\rho_{01} + \sqrt{\alpha^*\delta}\rho_{01}^* = r \tag{35}$$

for $r$ real. The intersection of the planes defines a line. $C_T$ remains constant on its intersection with the Bloch ball.

$C_T$ is zero if both terms in (30) vanish. Then the line segment cuts the Bloch sphere necessarily at pure states. That there are one or two pure states in the range of the channels (13) is proved in [19].

## 4. A special class of 1-qubit channels

We wish to extend the computations to some channels with more than two Kraus operators. It has been proved above that we can associate with every pair of operators, interpreted as Kraus operators, an antilinear Hermitian one,

$$\{A, B\} \longrightarrow \vartheta. \tag{36}$$

One may ask whether one can change the superoperator (1) without changing $\vartheta$ and, hence, without changing $C_T$ and $E_T$. To do so, we first observe that the trace-one condition is irrelevant for theorem 2. This fact simplifies the following a bit, and we can allow slightly more: after changing the Kraus operators, $\vartheta$, and hence $C_T$, may become scaled.

The answer is in the somehow surprising identity

$$(A \otimes B - B \otimes A)\,|\phi \otimes \phi\rangle = \langle\phi|\vartheta|\phi\rangle^*(|01\rangle - |10\rangle) \tag{37}$$

in which

$$|\phi \otimes \phi\rangle = x_0^2|00\rangle + x_0x_1(|01\rangle + |10\rangle) + x_1^2|11\rangle.$$

Consequently, if the superoperator $T'$ comes with Kraus operators $A'$ and $B'$, and if

$$A' = \mu_{11}A + \mu_{12}B \qquad B' = \mu_{21}A + \mu_{22}B \tag{38}$$

then the left-hand side of (37) changes by a factor only. The factor is the determinant of the transformation (38). Remembering the definition of $C_T$, it results that

$$C_{T'} = |\mu_{11}\mu_{22} - \mu_{01}\mu_{10}|\, C_T. \tag{39}$$

Now let us go a step farther and consider a channel

$$T'(\rho) = \sum_{j=1}^{m} A_j\rho A_j^\dagger. \tag{40}$$

For a small class of these channels, $C_T$ and, therefore, $E_T$ can be computed explicitly.

**Theorem 4.** *If the linear span of the Kraus operators $A_1, \ldots, A_m$ in (40) is at most two-dimensional, there is a unique antilinear and Hermitian $\vartheta'$ satisfying*

$$\det T'(\pi) = \operatorname{tr}\pi(\vartheta'\pi\vartheta') \tag{41}$$

*for pure $\pi$, and $C_{T'}$ is a flat roof.*

**Proof.** We use the identity

$$\det \sum \begin{pmatrix} a_ic_i & a_id_i \\ b_ic_i & b_id_i \end{pmatrix} = \sum_{j<k}(a_ib_k - a_kb_i)(c_id_k - c_kd_i) \tag{42}$$

to compute the determinant of $Y = \det T(\pi)$,

$$Y = \sum \begin{pmatrix} y_{i0}y_{i0}^* & y_{i0}y_{i1}^* \\ y_{i1}y_{i0}^* & y_{i1}y_{i1}^* \end{pmatrix} \tag{43}$$

where $\pi = |\phi\rangle\langle\phi|$, $\phi = x_0|0\rangle + x_1|\phi\rangle$,and

$$A_i\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_{i0} \\ y_{i1} \end{pmatrix}. \tag{44}$$

From (42) we obtain

$$\det T(\pi) = \sum_{j<k} |y_{0j}y_{1k} - y_{1j}y_{0k}|^2. \tag{45}$$

We choose $A$ and $B$ in (13) of the channel $T$ as linear generators of the linear span of the $A_j$ in (41). There are numbers $\mu_l^j$ fulfilling

$$A_j = \mu_1^jA + \mu_2^jB \qquad j = 1, \ldots, m \tag{46}$$

and allowing to rewrite

$$|y_{0j}y_{1k} - y_{1j}y_{0k}| = \left|\mu_1^j\mu_2^k - \mu_2^j\mu_1^k\right| \cdot |z_0w_1 - z_1w_0|\,.$$

With the help of (22) we finally obtain

$$\det T(\pi) = (\mathrm{tr}\, \pi \vartheta \pi \vartheta) \sum_{j<k} \left| \mu_1^j \mu_2^k - \mu_2^j \mu_1^k \right|^2 . \tag{47}$$

Hence, $\vartheta' = \mu\vartheta$, where $|\mu|^2$ can be read off from (47), does the job required by theorem 4. □

It seems that theorem 4 exhausts the possibilities to compute $C_T$ and $E_T$ by an antilinear and Hermitian $\vartheta$ for 1-qubit channels in the manner of the present paper. There are simple examples where the linear span of the Kraus operators is of dimension larger than two and for which one cannot find an appropriate $\vartheta$. For instance, the well known depolarizing channels

$$T_t(\rho) = [(\mathrm{tr}\, \rho)\mathbf{1} + s\rho](s + \dim \mathcal{H})^{-1}$$

which are positive for $-1 \leqslant s$ and completely positive for $-(\dim \mathcal{H})^{-1} \leqslant s$ belong to them.

The determinant of $T(\pi)$ is constant for pure states. Consequently, $C_T$ is constant everywhere and, trivially, a flat roof. But if this constant is different from zero, i.e. $s \neq 0$, it cannot be represented as (41) for all pure $\pi$ even if the dimension of $\mathcal{H}$ is two.

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